

On the pure state outcomes of Einstein-Podolsky-Rosen steering

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In the Einstein-Podolsky-Rosen experiment, when Alice makes a measurement on her part of a bipartite system, Bob's part is collapsed to, or steered to, a specific ensemble. Moreover, by reading her measurement outcome, Alice can specify which state in the ensemble Bob's system is steered to and with which probability. The possible conditional states that Alice can steer Bob's system to are called steering outcomes. In this work, we study the subset of steering outcomes which are pure after normalization. We illustrate that these pure steering outcomes, if they exist, often carry interesting information about the shared bipartite state. This information content becomes particularly clear when we study the purification of the shared state. Some applications are discussed. These include a generalisation of the fundamental lemma in the so-called 'all-versus-nothing proof of steerability' for systems with arbitrary dimensionality.

Introduction

Einstein-Podolsky-Rosen (EPR) steering, or just steering, indicates the ability of Alice to collapse Bob's system to a specific ensemble of states by measuring locally her own system, provided the two systems are in an appropriate quantum state. This was first discussed in the seminal papers of Einstein, Podolsky and Rosen [1] and Schrödinger [2]. Recently, Wiseman *et al.* gave a precise operational definition of steerability [3, 4], which distinguishes it from other classes of non-locality in quantum mechanics such as nonseparability [5] and Bell non-locality [6].

EPR steering can be seen also as a two-party quantum information processing task where Alice tries to convince Bob that the state they share is entangled. Bob does not trust Alice but he trusts on quantum mechanics and his own measurements [4]. This view has motivated a lot of research on how to witness steerability in a specific experimental setting by inequalities [7–13]. Beyond sufficient inequalities, a sufficient and necessary condition for steerability in terms of quantum channels has been developed [14, 15]. It has also been shown that the steerability is related to the concept of joint measurability of quantum observables [16–19].

On the other hand, some authors concentrate on characterizing the steering process in EPR experiments independently from a specific measurement setting. In this line, the EPR steering is characterized by a positive linear map, which maps a measurement outcome on Alice's system to a steering outcome on Bob's system [20]. Then an EPR experiment is analyzed based on all possible measurement outcomes on Alice's system, and all corresponding steering outcomes (conditional states) on Bob's system [20–23]. The concept of steering outcomes and its relatives have been proved useful in characteriz-

ing not only steerability but also more general aspects of bipartite quantum states [20, 21, 24, 25].

In this work we concentrate on special steering outcomes: those that are pure, or rank one, after normalization. In general, there may be no such pure steering outcomes. However, we show that if they exist, they often carry interesting information about the shared state. The special role of pure steering outcomes becomes particularly clear when we study the behaviour of the purified system in the EPR experiment. The following simple observation illustrates the idea. Suppose Charlie holds a third system such that the joint system of his with Alice and Bob is in a pure state. When Alice makes a measurement and obtains some outcome, the shared system of Bob and Charlie (BC) is steered to some state, which is nonseparable in general. However, if Alice learns that Bob's system is in a pure state, she knows immediately the two (BC) are separable and Charlie's system is also in a pure state. This is a clue that pure steering outcomes imply special structure of the shared state, which becomes visible in its purification. The detailed structure of the purified state is explained in the following sections. Applications in proving steerability are also discussed. Among these, a generalization of the fundamental lemma of the so-called 'all-versus-nothing proof of steerability' [26] is proved in an elementary way.

Measurement outcomes and steering outcomes

Consider the case where Alice and Bob share a state ρ of a bipartite quantum system over $\mathcal{H}_A \otimes \mathcal{H}_B$ with dimension $d_A \times d_B$. Suppose Alice makes a positive operator value measurement (POVM) $\{E_i\}_{i=1}^n$ on her system A , where the outcomes E_i are (semi-definite) positive operators such that $\sum_{i=1}^n E_i = \mathbb{I}_A$. Imagine that Alice can make all possible measurements. The set of all possible outcomes she can get consists of operators $\mathcal{M}_A = \{E | 0 \leq E \leq \mathbb{I}_A\}$, referred to as *measurement outcomes*. Regardless of the POVM she made, whenever Alice gets an outcome $E \in \mathcal{M}_A$, Bob's system B is

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steered to a conditional state $E' = \text{Tr}_A[\rho(E \otimes \mathbb{I}_B)]$. The set of possible conditional states Alice can steer Bob's system to is called the set of *steering outcomes* $\mathcal{M}'_A = \{E' | E \in \mathcal{M}_A\}$ [20, 23]. A steering outcome E' can also be characterized by the *steered state* $\sigma = E' / \text{Tr}(E')$ and its *steering probability* $p = \text{Tr}(E')$ [20, 21].

The set of steering outcomes and steered states have been studied recently and a deep connection to the non-local properties of the shared bipartite state has been revealed [20–23]. In this work, we consider a special subset of steered states, the *pure steered states*. We also use the term *pure steering outcomes* for those steering outcomes that give pure steered states after normalization. For a general bipartite state ρ , there may exist no pure steered states. However when they do exist, they often carry interesting information about the shared state ρ .

Pure steered states and projective measurements

To steer Bob's system to a specific state, we may imagine that Alice tries to design measurements that are efficient in the sense that the steering probability is maximal. We first note that to steer Bob's system to a pure state, projective measurements are as efficient as general POVMs.

Lemma 1. *If Alice can steer Bob's system to a pure state, she can always gain the maximal steering probability for that steered state with a projective measurement.*

Proof. Suppose Alice can steer Bob's system to $|\beta\rangle\langle\beta|$ with maximal probability p , then there exists a measurement outcome $E \in \mathcal{M}_A$ such that $E' = p|\beta\rangle\langle\beta|$. Let us consider the spectral decomposition of E , $E = \sum_{i=1}^n \lambda_i |\alpha_i\rangle\langle\alpha_i|$, where we kept only $\lambda_i > 0$. Accordingly, $E' = \sum_{i=1}^n \lambda_i (|\alpha_i\rangle\langle\alpha_i|)' = p|\beta\rangle\langle\beta|$. But since $|\beta\rangle\langle\beta|$ is a pure state, and pure states are extremal [27], this is only possible if for all i , either $(|\alpha_i\rangle\langle\alpha_i|)' = 0$ or $(|\alpha_i\rangle\langle\alpha_i|)' \propto |\beta\rangle\langle\beta|$. Suppose for $1 \leq i \leq m$, $(|\alpha_i\rangle\langle\alpha_i|)' \propto |\beta\rangle\langle\beta|$, and for $m \leq i \leq n$, $(|\alpha_i\rangle\langle\alpha_i|)' = 0$, then $\sum_{i=1}^m \lambda_i (|\alpha_i\rangle\langle\alpha_i|)' = p|\beta\rangle\langle\beta|$. This in fact implies that $\lambda_i = 1$ for $1 \leq i \leq m$, otherwise we can always increase λ_i to increase p . Thus $\sum_{i=1}^m (|\alpha_i\rangle\langle\alpha_i|)' = p|\beta\rangle\langle\beta|$. This means that for a projective measurement that contains all $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^m$, Alice is able to steer Bob's system to $|\beta\rangle\langle\beta|$ with probability p . Note that all measurement outcomes $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^m$ are mapped to the same steered state $|\beta\rangle\langle\beta|$, and the probability is accumulated to p . \square

A measurement outcome E is called a *projective measurement outcome* if it is a projection, i.e., $E = |\alpha\rangle\langle\alpha|$. In this case, the steering outcome $(|\alpha\rangle\langle\alpha|)'$ is called *projective steering outcome*. The above lemma implies that the maximal steering probability of a pure steered state can always be obtained by accumulating multiple projective steering outcomes.

Behaviour of the purified state in steering

Further insight can be gained by considering the purification of the joint state. Let C be an ancillary system attached to AB such that the whole system is in a pure state $|\Psi\rangle$ and $\rho = \text{Tr}_C(|\Psi\rangle\langle\Psi|)$. Since $|\Psi\rangle$ is pure, after Alice gets a projective measurement outcome on A , BC is also in a pure state. If further B is in a pure state, so must be C . This observation leads directly to the following lemma.

Lemma 2. *If with a projective measurement outcome $|\alpha\rangle\langle\alpha|$, Alice steers Bob's system to a pure state $|\beta\rangle\langle\beta|$ with probability p , then C is also collapsed to a pure state $|\gamma\rangle\langle\gamma|$. Accordingly, the purified state $|\Psi\rangle$ can be written as*

$$|\Psi\rangle = c|\alpha, \beta, \gamma\rangle + |\tilde{\Phi}\rangle, \quad (1)$$

where $|c|^2 = p$ and the partial projection of $|\tilde{\Phi}\rangle$ on $|\alpha\rangle$ vanishes, i.e., $\langle\alpha|\tilde{\Phi}\rangle = 0$. The tilde indicates that $|\tilde{\Phi}\rangle$ is not normalized.

Proof. Suppose Alice makes a projective measurement and gets $|\alpha\rangle\langle\alpha|$ with probability p , then the purified state can always be written as

$$|\Psi\rangle = c|\alpha\rangle|\delta\rangle + |\tilde{\Phi}\rangle, \quad (2)$$

where $|c|^2 = p$ and $\langle\alpha|\tilde{\Phi}\rangle = 0$. Here $|\delta\rangle$ is a pure state of BC . However if B is in pure state $|\beta\rangle$, we must have $|\delta\rangle = |\beta\rangle|\gamma\rangle$ and (1) follows. \square

Using Lemma 1, one can easily extend Lemma 2 to the case where the pure steering outcome $p|\beta\rangle\langle\beta|$ is an accumulation of several projective steering outcomes, $p|\beta\rangle\langle\beta| = \sum_{i=1}^m (|\alpha_i\rangle\langle\alpha_i|)'$. In this case, the purified state can be written as

$$|\Psi\rangle = \sum_{i=1}^m c_i |\alpha_i, \beta, \gamma_i\rangle + |\tilde{\Phi}\rangle, \quad (3)$$

where $\sum_{i=1}^m |c_i|^2 = p$ and $\langle\alpha_i|\tilde{\Phi}\rangle = 0$ for all i .

More interesting is the extension to the case of several different pure steering outcomes. In this case, we need to consider the relationship between them. The following lemma is an obvious extension when several pure steering outcomes are obtained in a single projective measurement.

Lemma 3. *If with a projective measurement $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^{d_A}$, Alice can steer Bob's system to (not necessarily orthogonal) pure states $\{|\beta_i\rangle\langle\beta_i|\}_{i=1}^{d_A}$ with probabilities $\{p_i\}_{i=1}^{d_A}$, then C is also collapsed to (not necessarily orthogonal) pure states $\{|\gamma_i\rangle\langle\gamma_i|\}_{i=1}^{d_A}$. Accordingly, the purified state $|\Psi\rangle$ can be written as*

$$|\Psi\rangle = \sum_{i=1}^{d_A} c_i |\alpha_i, \beta_i, \gamma_i\rangle. \quad (4)$$

with $|c_i|^2 = p_i$.

We ignore the proof, which is an easy extension of that of Lemma 2.

Example 1. Consider the case where Alice and Bob share two qubits in state

$$\rho = \eta|0, \beta_1\rangle\langle 0, \beta_1| + (1 - \eta)|1, \beta_2\rangle\langle 0, \beta_2| + \sqrt{\eta(1 - \eta)}(z|0, \beta_1\rangle\langle 1, \beta_2| + z^*|1, \beta_2\rangle\langle 0, \beta_1|), \quad (5)$$

where $0 \leq \eta \leq 1$, $|z| \leq 1$ and $|\beta_1\rangle$ and $|\beta_2\rangle$ are two arbitrary states. Note that for $z = 0$ the state ρ is separable and for $\eta = 1$ or $\eta = 0$ it is pure. The state can be purified by attaching a third qubit. The purified state is

$$|\Psi\rangle = \sqrt{\eta}|0, \beta_1, \gamma_1\rangle + \sqrt{1 - \eta}|1, \beta_2, \gamma_2\rangle, \quad (6)$$

with $\langle \gamma_2 | \gamma_1 \rangle = z$. This is of the form (4) and it is obvious that by measuring σ_z , Alice can steer Bob's system to pure states $|\beta_1\rangle$ or $|\beta_2\rangle$.

The form (4) of the purified state turns out to be very informative. Later, using Lemma 3, we will generalize the fundamental lemma of the all-versus-nothing proof of steerability, originally designed only for two qubits [26], to systems of arbitrary dimension (Theorem 2). Before doing so, we consider some other natural questions regarding the general properties of the pure steering outcomes.

The assumption in Lemma 3 requires that the steering outcomes are obtained in a single projective measurement. We ask the question if looking only at the steering outcomes, is it possible to establish that several pure steering outcomes are generated from a single projective measurement. We show that with a more restricted assumption, this is possible.

Lemma 4. *A set of distinguished projective steering outcomes $\{|\beta_i\rangle\langle\beta_i|\}_{i=1}^n$ which are orthogonal and which have steering probabilities $\{p_i\}_{i=1}^n$ summed up to 1 can be obtained in a single projective measurement.*

Proof. Suppose with projective measurement outcomes $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^n$, Alice can steer Bob's system to orthogonal states $\{|\beta_i\rangle\langle\beta_i|\}_{i=1}^n$ with probabilities $\{p_i\}_{i=1}^n$, $\sum_{i=1}^n p_i = 1$. We can assume that p_i are non-zero, since outcomes with empty probability can always be added in any measurement. We are going to show that $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^n$ are orthogonal. This would imply that we can extend $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^n$ to form a projective measurement.

According to Lemma 2, for each measurement outcome $|\alpha_i\rangle\langle\alpha_i|$ we can write

$$|\Psi\rangle = c_i|\alpha_i, \beta_i, \gamma_i\rangle + |\tilde{\Phi}_i\rangle, \quad (7)$$

where $\langle\alpha_i|\tilde{\Phi}_i\rangle = 0$ with $|c_i|^2 = p_i$. Note that in the Hilbert space of ABC , the set $\{|\alpha_i, \beta_i, \gamma_i\rangle\}_{i=1}^n$ are orthogonal because of the orthogonality of $\{|\beta_i\rangle\}_{i=1}^n$. Moreover $\sum_{i=1}^n |c_i|^2 = 1$. It then follows that

$$|\Psi\rangle = \sum_{i=1}^n c_i|\alpha_i, \beta_i, \gamma_i\rangle, \quad (8)$$

by exhaustive expansion of $|\Psi\rangle$ with respect to orthonormal states $\{|\alpha_i, \beta_i, \gamma_i\rangle\}_{i=1}^n$. Comparing with (7), we get

$$|\tilde{\Phi}_i\rangle = \sum_{j=1, j \neq i}^n c_j|\alpha_j, \beta_j, \gamma_j\rangle. \quad (9)$$

Since $\langle\alpha_i|\tilde{\Phi}_i\rangle = 0$, we have

$$0 = \sum_{j=1, j \neq i}^n c_j \langle\alpha_i|\alpha_j\rangle |\beta_j, \gamma_j\rangle. \quad (10)$$

But since $\{|\beta_j, \gamma_j\rangle\}_{j=1, j \neq i}^n$ are orthogonal and $c_j \neq 0$, we must have $\langle\alpha_i|\alpha_j\rangle = 0$ for $j \neq i$, i.e., $\{|\alpha_i\rangle\langle\alpha_i|\}_{i=1}^n$ are orthogonal. \square

Example 2. One can consider the state (5) over two qubits, now with $|\beta_1\rangle = |0\rangle$ and $|\beta_2\rangle = |1\rangle$. The two projective steering outcomes $(|0\rangle\langle 0|)' = \eta|0\rangle\langle 0|$ and $(|1\rangle\langle 1|)' = (1 - \eta)|1\rangle\langle 1|$ are orthogonal and have probabilities summed up to 1. Therefore, they must be obtainable from a single projective measurement on A , which is σ_z in this case. It is also clear from this example that the assumption in Lemma 4, which requires $\langle\beta_1|\beta_2\rangle = 0$, is more restrictive than the conclusion of Lemma 3.

The emergence of subspaces of pure steered states

Suppose Alice and Bob share two qubits which are in a pure nonseparable state. When Alice obtains a projective outcome in some measurement, Bob's system is steered to a pure state. With all possible projective outcomes, Alice can steer Bob's system to all possible pure states with different probabilities. We see that Alice's pure steered states cover the whole Hilbert space of Bob's system.

We will see that in general Alice's steered states cover only a subspace of Bob's Hilbert space. This notion of subspace of pure steered states also appears when the shared bipartite system is in a suitable mixed state. In fact, these mixed states can be viewed as 'partial pure states'—states that look like pure ones in some restricted subspace. The following theorem allows us to characterize these particular mixed states via their behaviour in EPR experiments.

Theorem 1. *Suppose the set of Alice's steering outcomes contains a subset of n projective steering outcomes $\{p_i|\beta_i\rangle\langle\beta_i|\}_{i=1}^n$ with probabilities p_i summed up to 1. Moreover, suppose Alice can steer Bob's system to another pure state $|\beta\rangle$, then she can also steer Bob's system to any state in $\text{span}(\{|\beta_i\rangle\langle\beta_i|\}_{i=1}^n, |\beta\rangle\langle\beta|)$.*

Note that here and from now on we also use vectors such as $|\beta\rangle$ on behalf of their projections $|\beta\rangle\langle\beta|$ to indicate pure states.

Example 3. When applied to a two-qubit system, this discussion is even simpler. Suppose by projective measurements, Alice can steer Bob's system to two orthogonal

pure states with probabilities summed up to 1, and another additional state, then AB must be in a pure state. This can be an operational way for Alice to prove to Bob that their shared state is a pure entangled one (instead of steering Bob's system to infinitely many pure states).

In order to prove this theorem, we need the following simple lemma.

Lemma 5. *Let $\{|\alpha_i\rangle\}_{i=1}^n$ be a set of $n \leq d_A$ orthonormal states of system A , and $\{|\beta_i\rangle\}_{i=1}^n$ be arbitrary states of system B . If there exist n non-zero numbers a_i such that the linear combination $\sum_{i=1}^n a_i |\alpha_i\rangle |\beta_i\rangle$ is a product state, then $|\beta_1\rangle = |\beta_2\rangle = \dots = |\beta_n\rangle$.*

Proof. Suppose $\sum_{i=1}^n a_i |\alpha_i\rangle |\beta_i\rangle$ is a product state, i.e., $\sum_{i=1}^n a_i |\alpha_i\rangle |\beta_i\rangle = |\alpha\rangle |\beta\rangle$. Then when tracing over A , one gets

$$|\beta\rangle\langle\beta| = \sum_{i=1}^n a_i a_i^* |\beta_i\rangle\langle\beta_i|, \quad (11)$$

where we have used the fact that $\{|\alpha_i\rangle\}_{i=1}^n$ are orthogonal. Since pure states are extremal states [27], this only happens for non-zero a_i if $|\beta_1\rangle = |\beta_2\rangle = \dots = |\beta_n\rangle$. \square

Proof of theorem 1. Let us attach a system C to purify AB to a state $|\Psi\rangle$. Because of lemma 4, the purified state takes the form

$$|\Psi\rangle = \sum_{i=1}^n c_i |\alpha_i, \beta_i, \gamma_i\rangle, \quad (12)$$

where $\{|\alpha_i\rangle\}_{i=1}^n$ are orthogonal. Now if Alice can steer B to another state $|\beta\rangle$ by a projective measurement, by Lemma 2, there exists state $|\alpha\rangle$ in \mathcal{H}_A and state $|\gamma\rangle$ in \mathcal{H}_C such that $\langle\alpha|\Psi\rangle = c|\beta\rangle|\gamma\rangle$ for some $c \neq 0$. Using (12), one has

$$|\beta, \gamma\rangle = \sum_{i=1}^n \frac{c_i \langle\alpha|\alpha_i\rangle}{c} |\beta_i, \gamma_i\rangle. \quad (13)$$

Since $\{|\beta_i\rangle\}_{i=1}^n$ are orthogonal by assumption, one finds $\langle\beta_i|\beta\rangle = \frac{c_i}{c} \langle\alpha|\alpha_i\rangle \langle\gamma|\gamma_i\rangle$. For any i such that $\langle\beta_i|\beta\rangle \neq 0$, one has $\frac{c_i}{c} \langle\alpha|\alpha_i\rangle \neq 0$. Thus by Lemma 5, (13) implies that for all $|\beta_i\rangle$ such that $\langle\beta_i|\beta\rangle \neq 0$, $|\gamma_i\rangle = |\gamma\rangle$. Without loss of generality, we assume that $\langle\beta_i|\beta\rangle \neq 0$ for $1 \leq i \leq m$ and $\langle\beta_i|\beta\rangle = 0$ for $m < i \leq n$. The purified state $|\Psi\rangle$ can now be written as

$$|\Psi\rangle = \left(\sum_{i=1}^m c_i |\alpha_i, \beta_i\rangle \right) |\gamma\rangle + \sum_{i=m+1}^n c_i |\alpha_i, \beta_i, \gamma_i\rangle. \quad (14)$$

Note that $\{|\alpha_i\rangle\}_{i=1}^m$ are orthogonal due to Lemma 4. We claim that with this form, the state allows Alice to steer B to arbitrary state in $\text{span}(\{|\beta_i\rangle\}_{i=1}^m)$. To see that we just calculate the steering outcomes for an arbitrary projective measurement outcome $|\alpha\rangle = \sum_{i=1}^m a_i |\alpha_i\rangle$,

$$\langle\alpha|\Psi\rangle = \left(\sum_{i=1}^m a_i^* c_i |\beta_i\rangle \right) |\gamma\rangle, \quad (15)$$

where the separated ancillary state $|\gamma\rangle$ can be simply ignored. Since $c_i \neq 0$ and a_i are arbitrary, the projective measurement outcomes (15) cover the whole $\text{span}(\{|\beta_i\rangle\}_{i=1}^m)$. \square

Example 4. Consider the case where Alice and Bob share two qutrits in state

$$\rho = \eta |\psi^-\rangle\langle\psi^-| + (1-\eta) |0,0\rangle\langle 0,0| + \sqrt{\eta(1-\eta)} (z |\psi^-\rangle\langle 0,0| + z^* |0,0\rangle\langle\psi^-|), \quad (16)$$

with $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|+1, -1\rangle - |-1, +1\rangle)$, $0 < \eta < 1$ and $|z| \leq 1$. The state can be purified by attaching a qubit. The purified state is

$$|\Psi\rangle = \sqrt{\eta} |\psi^-\rangle |\gamma_1\rangle + \sqrt{1-\eta} |0,0\rangle |\gamma_2\rangle, \quad (17)$$

with $\langle\gamma_2|\gamma_1\rangle = z$, which is clearly of the form (14). It is then easy to see that Alice can steer Bob system to any linear combination of $|-1\rangle$ and $|+1\rangle$.

Pure steering outcomes and steerability

In their paper, Wiseman *et al.* [3] discovered that only for certain states the EPR steering experiment is verifiable to be truly nonlocal, while for other states the steering experiment can actually be locally simulated. The former are regarded as *steerable* states, and the latter are called *unsteerable* states.

To demonstrate steering, Alice prepares a bipartite quantum state ρ over $\mathcal{H}_A \otimes \mathcal{H}_B$. She sends part B to Bob and specifies a set of measurements A (called *measurement assemblage*) she can make. Bob then asks her to make a measurement A_x from A , which consists of outcomes $\{A_{a|x}\}_a$. As we described above, it is expected that upon Alice making the measurement, Bob's system is steered to normalized conditional states $A'_x = \{\text{Tr}_A[\rho(A_{a|x} \otimes \mathbb{I})]\}_a$, which form an ensemble $\{P(a|x), \rho_{a|x}\}_a$ with $P(a|x) = \text{Tr}(A'_{a|x})$ and $\rho_{a|x} = A'_{a|x}/P(a|x)$. When receiving the outcome from Alice, Bob can perform state tomography to verify his corresponding conditional state. Being able to perform any specified measurement x , Alice intends to convince Bob that she can steer his system to different ensembles in the steering assemblage A' from a distance.

However, when Bob does not trust Alice, then this verification protocol may not be always convincing. Indeed, there exists a strategy for Alice to cheat Bob. In the cheating strategy, instead of sending Bob halves of entangled systems, Alice sends him random states choosing from an ensemble of *Local Hidden States (LHS)* $\{P(\xi), \sigma_\xi\}_\xi$. Upon receiving the request of measuring x , she simulates the measurement outcomes by randomly choosing an outcome a with a designed probability function $P(a|x, \xi)$. Such a simulation gives exactly the expected result of state tomography by Bob if

$$A'_{a|x} = \sum_{\xi} P(a|x, \xi) P(\xi) \sigma_\xi, \quad (18)$$

in which case the state is *unsteerable* (here always considered from Alice's side) with measurement assemblage A . It is clear that the existence of the LHS ensemble, which allows for this cheating strategy, depends crucially on the steering assemblage A' . The steering assemblage A' , in turn, depends on both the shared state ρ and the measurement assemblage A . In the following, if the measurement assemblage is not explicitly specified, we assume that it consists of all projective measurements.

Following the above consideration, we call $\mathcal{A} = \{A_{a|x}\}_{a,x}$ the set of measurement outcomes for the assemblage A , which is a subset of \mathcal{M}_A . Accordingly, $\mathcal{A}' = \{A'_{a|x}\}_{a,x}$ is a subset of \mathcal{M}'_A , called the set of steering outcomes for the assemblage A . The problem of determining the existence of a LHS ensemble is also greatly simplified if \mathcal{A}' contains one or several (unnormalised) pure states. At the heart of this simplification is the following lemma.

Lemma 6. *If Alice can steer Bob's system to a pure state $|\beta\rangle\langle\beta|$ with probability p , then the LHS ensemble, if exists, must contain $|\beta\rangle\langle\beta|$ with probability no less than p .*

Proof. By assumption, in equation (18), $A'_{a|x} = p|\beta\rangle\langle\beta|$ for some a and x . But since pure states are extremal points of the set of states of B [27], the decomposition in terms of states of the LHS ensemble (18) is only possible if $\sigma_{\xi_0} = |\beta\rangle\langle\beta|$ for some ξ_0 and $P(a|x, \xi)$ all vanish except for $\xi = \xi_0$. Moreover $P(a|x, \xi_0) \leq 1$, thus $P(\xi_0) \geq p$. \square

Despite its simplicity, Lemma 6 is in fact very useful. As a simple corollary, if Alice can steer Bob to some pure states (not necessarily orthogonal) with probabilities summed up greater than 1, then they cannot be contained in any LHS ensemble and the state of AB is steerable from Alice's side.

Example 5. For a pure nonseparable state, Alice can steer Bob's system to infinitely many pure states by using different projective measurements. The total probability is infinite, therefore nonseparable pure states are all steerable.

Example 6. One can take the state (16) in example 4 with $0 < \eta < 1$. Although the state is mixed, the subspace of pure steered state is infinite. Steerability is therefore also guaranteed. Although we have obtained the results with fairly elementary arguments, it is interesting to note that the steerability of the state is not easily detected by steering inequalities. For example, consider the steering inequality proposed in [28],

$$|\langle S^{SA} \otimes S^{SB} \rangle|^2 > \langle ((S^x)^2 + (S^y)^2 - \frac{7}{16}) \otimes ((S^x)^2 + (S^y)^2) \rangle, \quad (19)$$

where $\langle K_{AB} \rangle = \text{tr}[K_{AB}\rho_{AB}]$, $S^{SA/B}$ with $s_A, s_B = \pm$ are the spin raising and lowering operators, and S^i with $i = x, y, z$ are the usual self-adjoint spin operators. In the inequality, s_A and s_B can be chosen independently

to optimize the possible violation. When the inequality is true, it witnesses the steerability of the state. For simplicity, we consider a simple case of (16) where $z = 0$,

$$\rho = \eta|\psi^-\rangle\langle\psi^-| + (1-\eta)|00\rangle\langle 00|. \quad (20)$$

For this state, the left hand-side of equation (19) gives 0 and the right hand-side gives $\frac{1}{16}(50 - 41\eta)$. Thus the inequality is not violated. Note also that this example can also be easily extended to systems of arbitrary dimension.

Following this line of argument, when the steering outcomes \mathcal{A}' contain a set of pure states with steering probabilities summed up to 1, the LHS ensemble, if exists, can only be these pure states with their corresponding steering probabilities. Determining the steerability of the state is therefore reduced to checking if the LHS ensemble can explain all measurements in the assemblage A in the sense of (18). The following theorem utilizes the idea.

Theorem 2. *If by a projective measurement Alice can steer Bob's system B to a set of independent pure states, then the joint state of AB is either separable or steerable.*

Lemma 7. *Let $\{|\beta_i\rangle\}_{i=1}^n$ be independent states of some quantum system, then $\{E_{ij} = |\beta_i\rangle\langle\beta_j|\}_{i,j=1}^n$ are independent operators.*

Proof. We can suppose that $\{|\beta_i\rangle\}_{i=1}^n$ is a basis of the system; otherwise we can extend it to form a basis. Let $\{|\beta'_j\rangle\}_{j=1}^n$ be the dual basis, defined by $\langle\beta'_i|\beta_j\rangle = \delta_{ij}$. Consider a linear combination $Q = \sum_{i,j=1}^n \lambda_{ij} E_{ij}$, we observe that $\langle\beta'_i|Q|\beta'_j\rangle = \lambda_{ij}$. It follows that $Q = 0$ if and only if $\lambda_{ij} = 0$ for all i and j , or $\{E_{ij}\}_{i,j=1}^n$ are independent. \square

Proof of theorem 2. Suppose the joint state is unsteerable, we will show that it is separable. Suppose Alice performs the projective measurement $A_x = \{|\alpha_{a|x}\rangle\langle\alpha_{a|x}|\}_{a=1}^{d_A}$ locally. After the measurement, Alice's state is described by an ensemble $\{P(a|x), |\alpha_{a|x}\rangle\langle\alpha_{a|x}|\}_{a=1}^{d_A}$ and correspondingly, Bob's system is steered to an ensemble of not necessarily orthogonal pure states $\{P(a|x), |\beta_{a|x}\rangle\langle\beta_{a|x}|\}_{a=1}^{d_A}$. Note that $\sum_a P(a|x) = 1$. It follows from our above discussion that the LHS ensemble must be $\{P(a|x), |\beta_{a|x}\rangle\langle\beta_{a|x}|\}_{a=1}^{d_A}$. Obviously, we have a very important information.

Let us attach a system C to purify the state ρ of AB to a pure state $|\Psi\rangle$ of ABC . From Lemma 3, we know that the purified state must be of the form

$$|\Psi\rangle = \sum_{a=1}^{d_A} c_{a|x} |\alpha_{a|x}\rangle |\beta_{a|x}\rangle |\gamma_{a|x}\rangle. \quad (21)$$

with $|c_{a|x}|^2 = P(a|x)$. Then let Alice make another measurement $A_{x'}$ in the assemblage A (here being all possible projective measurements), which is a collection of new projections $\{A_{a'|x'} = |\alpha_{a'|x'}\rangle\langle\alpha_{a'|x'}|\}_{a'=1}^{d_A}$. The new projections are related to $\{|\alpha_{a|x}\rangle\langle\alpha_{a|x}|\}_{a=1}^{d_A}$ by a $d_A \times d_A$

unitary matrix U ,

$$|\alpha_{a'}|_{x'}\rangle = \sum_{a=1}^{d_A} |\alpha_a|_x\rangle U_{aa'}. \quad (22)$$

Now the joint state can be expressed using $\{|\alpha_{a'}|_{x'}\rangle\}_{a'=1}^{d_A}$ as

$$|\Psi\rangle = \sum_{a=1}^{d_A} \sum_{a'=1}^{d_A} c_{a|x} U_{aa'}^* |\alpha_{a'}|_{x'}\rangle |\beta_a|_x\rangle |\gamma_a|_x\rangle. \quad (23)$$

If in the measurement x' , Alice gets the outcome a' , BC is then steered to

$$\sum_{a=1}^{d_A} c_{a|x} U_{aa'}^* |\beta_a|_x\rangle |\gamma_a|_x\rangle. \quad (24)$$

After tracing out the ancillary system C , one gets the corresponding conditional state of B ,

$$A'_{a'|x'} = \sum_{b,d=1}^{d_A} c_{b|x} c_{d|x}^* U_{ba'}^* U_{da'} \langle \gamma_d|x| \gamma_b|x \rangle |\beta_b|_x\rangle \langle \beta_d|x|. \quad (25)$$

Note that $\{|\beta_a|_x\rangle\}_{a=1}^{d_A}$ are independent, thus so are $\{|\beta_b|_x\rangle \langle \beta_d|x|\}_{b,d=1}^{d_A}$ by Lemma 7. The decomposition (25) is therefore unique.

Since the $\{P(a|x), |\beta_a|_x\rangle \langle \beta_a|x|\}_{a=1}^{d_A}$ is the LHS ensemble, the conditional states $A'_{a'|x'}$ must be expanded by projections $\{|\beta_a|_x\rangle \langle \beta_a|x|\}_{b=1}^{d_A}$ for all unitary matrices U and for all outcomes a' . This means those operators $|\beta_b|_x\rangle \langle \beta_d|x|$ with $b \neq d$ must be absent from the decomposition (25), i.e.,

$$c_{b|x} c_{d|x}^* U_{ba'}^* U_{da'} \langle \gamma_d|x| \gamma_b|x \rangle = 0. \quad (26)$$

Now for a particular pair $b \neq d$, one can choose some a' and a unitary matrix U such that $U_{ba'}^* U_{da'} \neq 0$. (For example, one can choose $a' = b$ and the matrix U such that $U_{bb} = U_{bd} = 1/\sqrt{2} = U_{dd} = -U_{db}$ and identity blocks everywhere else.) This means that (26) is satisfied only when $c_{b|x} c_{d|x}^* \langle \gamma_d|x| \gamma_b|x \rangle = 0$ for all $b \neq d$. This makes the state of AB , $\rho = \text{Tr}_C(|\Psi\rangle \langle \Psi|)$, manifestly separable: $\rho = \sum_{b,d=1}^{d_A} c_{b|x} c_{d|x}^* \langle \gamma_d|x| \gamma_b|x \rangle |\alpha_b|_x\rangle \langle \alpha_d|x| = \sum_{b=1}^{d_A} |c_{b|x}|^2 |\alpha_b|_x\rangle \langle \alpha_b|x|$. \square

In fact, from this proof, a more detailed statement can be made.

Corollary. Suppose by making a projective measurement consisting of orthonormal states $\{|\alpha_i\rangle\}_{i=1}^{d_A}$, Alice can steer Bob system to independent pure states $\{|\beta_i\rangle\}_{i=1}^{d_A}$, then either:

- (i) the joint state is steerable (thus nonseparable), in which case $\langle \alpha_i, \beta_i | \rho | \alpha_j, \beta_j \rangle$ must not vanish for some $j \neq i$;
- (ii) the joint state ρ is separable of the form $\rho = \sum_{i=1}^{d_A} p_i |\alpha_i, \beta_i\rangle \langle \alpha_i, \beta_i|$.

Proof. Again applying Lemma 3, the assumption of the corollary implies that the purified state is of the form (21),

$$|\Psi\rangle = \sum_{i=1}^{d_A} c_i |\alpha_i, \beta_i, \gamma_i\rangle. \quad (27)$$

By tracing over the ancillary system, we find

$$\rho = \sum_{i=1, j=1}^{d_A} c_i c_j^* \langle \gamma_j | \gamma_i \rangle |\alpha_i, \beta_i\rangle \langle \alpha_j, \beta_j|. \quad (28)$$

Note that $\langle \alpha_i, \beta_i | \rho | \alpha_j, \beta_j \rangle = c_i c_j^* \langle \gamma_j | \gamma_i \rangle$. Then it is clear that if $\langle \alpha_i, \beta_i | \rho | \alpha_j, \beta_j \rangle \neq 0$ for some $j \neq i$, the state is steerable (thus nonseparable) according to the above proof; else if $\langle \alpha_i, \beta_i | \rho | \alpha_j, \beta_j \rangle = 0$ for all $j \neq i$, ρ reduces to the separable form stated in (ii) with $p_i = |c_i|^2$. \square

When applied to a two-qubit system, the statement is particularly simple. For a two-qubit system, if by a projective measurement Alice can steer B to two pure states, then the joint state of AB is either separable or steerable. This result has been recently obtained by Chen *et al.* [26]. Experimental applications of this special case were also discussed [29]. Here we report a more general result with a somewhat more systematic proof.

Example 7. Take again the two-qubit state (5) in example 1, which satisfies the assumption of the above corollary. Note that $\langle 0, \beta_1 | \rho | 1, \beta_2 \rangle = z \sqrt{\eta(1-\eta)}$. We then know that it is steerable (or equivalently, separable) if and only if $|z|^2 \eta(1-\eta) > 0$.

Conclusion

We have shown that the pure state outcomes in the EPR experiments carry interesting information about the shared state, in particular its nonlocal properties. Thanks to the purification technique, results are obtained in an elementary and systematic way. Although our work only concentrates on the pure steering outcomes, we speculate that analysing the behaviour of the purified system in the EPR experiments might be an interesting approach to quantum nonlocality. Applications of the special case of Theorem 2 for two-qubit systems have been discussed [26, 29]. As experimental research is moving toward higher dimensional systems, we hope that our generalization will be useful for the future experiments.

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